Model reduction: Balanced truncation

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Outline

- Description of problem
- Discrete time systems
- Controllability
- Observability
- Change of basis
- Balanced representations
- How to obtain balanced representations?
- Balanced truncation
  - Error bounds
  - Stability
- ERA

Description of problem

Non-linear dynamics with one input and one output:

$$\begin{cases} \partial_t w = \mathcal{A}(w) + \mathcal{B}(u) \\ y = \mathcal{C}(w) \end{cases}$$

Fixed point:

$$\mathcal{A}(w_0) + \mathcal{B}(0) = 0$$

Dynamics around fixed point:

$$\begin{cases} w(t) := w_0 + \varepsilon w(t) \\ u(t) := 0 + \varepsilon u(t) \\ y(t) \coloneqq \mathcal{C}(w_0) + \epsilon y(t) \end{cases}$$

Linearization:

$$\begin{cases} \varepsilon \partial_t w = \mathcal{A}(w_0) + \varepsilon A w + \mathcal{B}(0) + \varepsilon B u \\ \mathcal{C}(w_0) + \epsilon y(t) = \mathcal{C}(w_0) + \varepsilon C w \end{cases}$$

Linear-Time-Invariant model:

$$\begin{cases} \partial_t w = A w + B u \\ y = C w \end{cases}$$

Description of problem

- Continuous model (PDE): $\partial_t w$

  $\downarrow$ Spatial discretization
- Large-dimensional model obtained after spatial discretization (ODE): $d_t w$

  $\downarrow$ Model reduction
- Reduced-order model (ODE)

Description of problem

Let us consider the following large-scale (dimension n) single-input single-output (SISO) problem:

$$d_t w = Aw + Bu$$
$$y = Cw$$

where A is stable.

<u>Solution:</u>

$$y(t) = Ce^{A(t-t_0)}w_I + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau$$

If $t_0 \to -\infty$ and $w_I = 0$:

$$y(t) = \int_{-\infty}^{t} Z(t-\tau)u(\tau)d\tau$$

where:

$$Z(t) = Ce^{At}B, t \geq 0$$

Dynamics from input to output is fully characterized by $Z(t)$

Description of problem: impulse response

$Z(t)$ corresponds to the impulse response of the system.
The impulse response of the system is defined as follows: let us consider the system driven by $u(t) = \delta(t)$, with the initial condition $\mathrm{w}(0^-) = w_I = 0$. The solution is given by:

$$\Rightarrow y(t) = Ce^{At}B = Z(t) \text{ for } t \geq 0.$$

Proof:

$$\int_{0^-}^{0^+} (d_t w)dt = \int_{0^-}^{0^+} (Aw + Bu)dt$$

$$[w]_{0^-}^{0^+} = 0 + B\int_{0^-}^{0^+} \big(\delta(t)\big)dt$$

$$w(0^+) - w_I = B$$

Finally:

$$w(t) = e^{At}B \text{ and } y(t) = Ce^{At}B$$

Description of problem: transfer function

The transfer function $T_{yu}(i\omega)$ between u and y is obtained by considering the frequency domain: $u(t) = e^{i\omega t}\hat{u}, w(t) = e^{i\omega t}\widehat{w}, y(t) = e^{i\omega t}\hat{y}$. It corresponds to the ratio of $\hat{y}$ and $\hat{u}$:

$$T_{yu}(i\omega) = \frac{\hat{y}}{\hat{u}} = C(i\omega I - A)^{-1}B$$

It may be shown that $T_{yu}(i\omega)$ is equal to the Fourier transform of $Z(t)$:

$$T_{yu}(i\omega) = \widehat{Z}\,(i\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} Z(t)\,dt$$

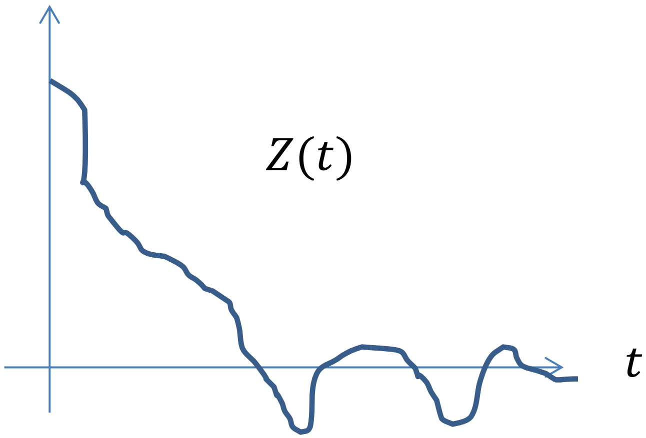
<u>Proof:</u>
An eigenvalue decomposition of $A = V\Lambda V^{-1}$ yields:

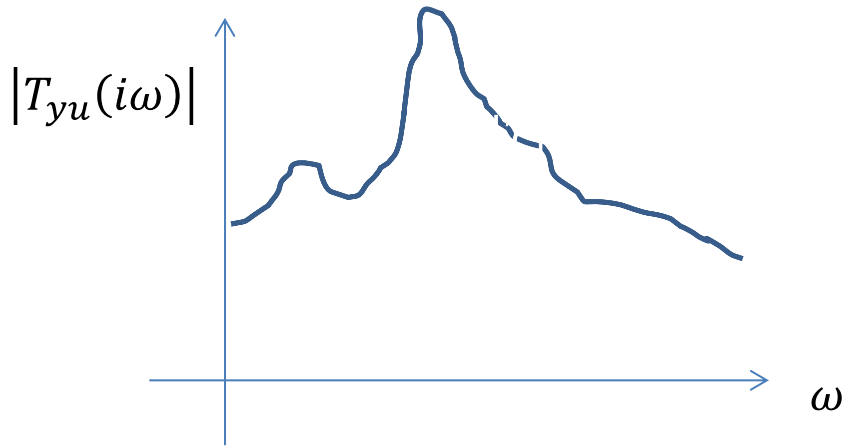
$$Z(t) = Ce^{At}B = CVe^{\Lambda t}V^{-1}B$$

Therefore:

$$\widehat{Z}\,(i\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} Z(t)\,dt = \int_{0}^{+\infty} e^{-i\omega t} Z(t)\,dt = CV\left[\int_{0}^{+\infty} e^{(\Lambda - i\omega I)t}\,dt\right]V^{-1}B =$$
$$CV\left[(\Lambda - i\omega I)^{-1}e^{(\Lambda - i\omega I)t}\right]_{0}^{+\infty}V^{-1}B = -CV(\Lambda - i\omega I)^{-1}V^{-1}B = C(i\omega I - A)^{-1}B = T_{yu}(i\omega)$$

Description of problem





Reduced-Order-Modelling consists in finding a small-scale system:

$$\begin{cases} d_t w_r = A_r w_r + B_r u \\ y = C_r w_r \end{cases}$$

which preserves these curves.
In particular,

$$C e^{At} B \approx C_r e^{A_r t} B_r \ \forall t \geq 0$$
$$C(i\omega I - A)^{-1} B \approx C_r (i\omega I - A_r)^{-1} B_r \ \forall \omega$$

Description of problem

What does preserve mean ?

Let us introduce the following norms to quantify the input-output relation:
1/ The 2-norm:

$$\|Z\|_2 = \sqrt{\int_0^{+\infty} Z(t)^2 dt} \underbrace{=}_{\text{Parseval}} \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|T_{yu}(i\omega)\right|^2 d\omega}$$

2/ The ∞-norm:

$$\|Z\|_\infty = \max_{\omega} \left|T_{yu}(i\omega)\right| = \max_{u(t)} \frac{\sqrt{\int_0^{+\infty} y(t)^2 dt}}{\sqrt{\int_0^{+\infty} u(t)^2 dt}}$$

Preserve means that we should find Z_r such that $\|Z - Z_r\|$ is minimal in one of the chosen norms.

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Discrete time systems

Discrete time system:

$$w(k+1) = Aw(k) + Bu(k)$$
$$y(k) = Cw(k)$$

Link between continuous time and discrete time systems:

Integration of linearized Navier-Stokes equations over sampling time Δt (with control signal constant over t and $t + \Delta t$)

$$w(t+\Delta t) = \underbrace{e^{A\Delta t}}_{\text{new}A} w(t) + \underbrace{\int_0^{\Delta t} e^{A(\Delta t - \tau)} B d\tau}_{\text{new } B} u(t)$$

Note that the sampling time Δt may be large!

Solution (for $w(0) = 0$):

$$y(k) = \sum_{j=0}^{k-1} Z(k-j)u(j)$$

Where $Z(k)$ designates the impulse response of the system (A, B, C):

$$Z(k) = CA^{k-1}B, k \geq 1$$

Discrete time systems

<u>Impulse response:</u>
Z_k correponds to the solution with the following driving terms:

$$u(0) = 1, u(k \geq 1) = 0$$
$$w(0) = 0$$

Then:

$$k = 0, u = 1, w = 0, y = 0$$
$$k = 1, u = 0, w = B, y = CB$$
$$k = 2, u = 0, w = AB, y = CAB$$
$$k = 3, u = 0, w = A^2B, y = CA^2B$$

$$\Rightarrow k, u = 0, w = A^{k-1}B, y = CA^{k-1}B$$

Discrete time systems

<u>Transfer function:</u>

Let us consider a mode, an excitation and a measurement of period n (which corresponds to a time-period $T = n\Delta t \Rightarrow \omega = \frac{2\pi}{n\Delta t}$):

$$u(k) = e^{\frac{2\pi i k}{n}}\hat{u}, w(k) = e^{\frac{2\pi i k}{n}}\widehat{w}, y(k) = e^{\frac{2\pi i k}{n}}\hat{y}$$

Introducing the notation $z = e^{\frac{2\pi i}{n}} (= e^{i\omega\Delta t})$, the transfer function from u to y is :

$$T_{yu}(z) = \frac{\hat{y}}{\hat{u}} = C(zI - A)^{-1}B$$

<u>Proof:</u>

$$w(k+1) = Aw(k) + Bu(k) \Rightarrow \overbrace{e^{\frac{2\pi i}{n}}}^{z}\widehat{w} = A\widehat{w} + \mathrm{B}\hat{\mathrm{u}}$$
$$\Rightarrow \widehat{w} = (zI - A)^{-1}B\hat{u}$$

It may be shown that:

$$T_{yu}(z) = Z(1)z^{-1} + Z(2)\, z^{-2} + Z(3)z^{-3} + Z(4)z^{-4} + \cdots$$

where: $Z(k) = CA^{k-1}B$ is the impulse response of the system.

$$T_{yu}(z) = \frac{\hat{y}}{\hat{u}} = C(zI - A)^{-1}B$$
$$= Cz^{-1}(I + Az^{-1} + A^2z^{-2} + A^3z^{-3} + \cdots)B \text{ (since } A \text{ is stable)}$$
$$= CBz^{-1} + CABz^{-2} + CA^2Bz^{-3} + CA^3Bz^{-4} + \cdots$$
$$\approx Z(1)z^{-1} + Z(2)\, z^{-2} + Z(3)z^{-3} + Z(4)z^{-4} + \cdots$$

Discrete time systems

<u>Norms of system (A, B, C) in discrete time:</u>

1/ 2-norm:

$$||Z||_2 = \sqrt{\sum_{k=0}^{\infty} Z(k)^2}$$

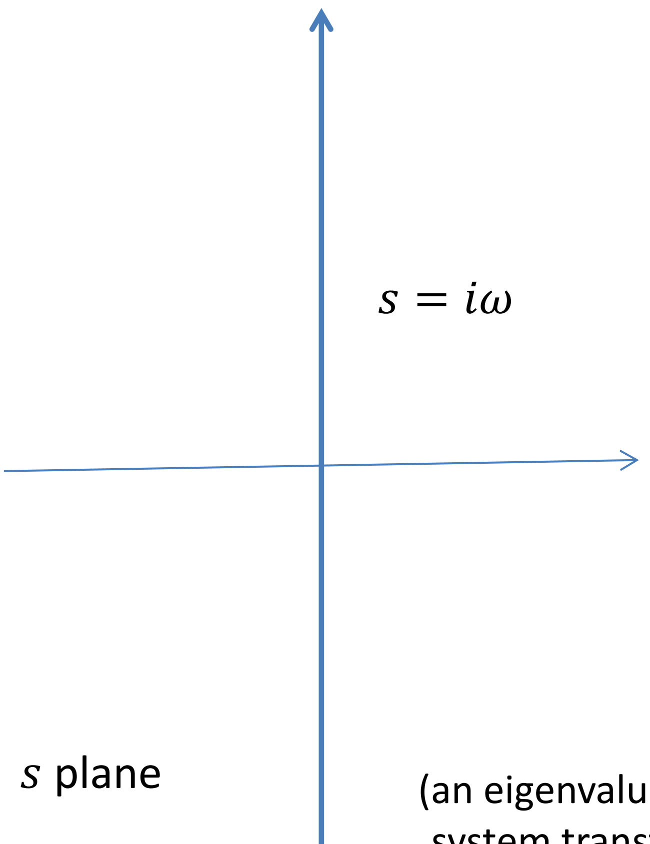
2/ ∞-norm:

$$||Z||_\infty = \max_{z=e^{\frac{2\pi i}{n}}, n\geq 1} |T_{yu}(z)|$$

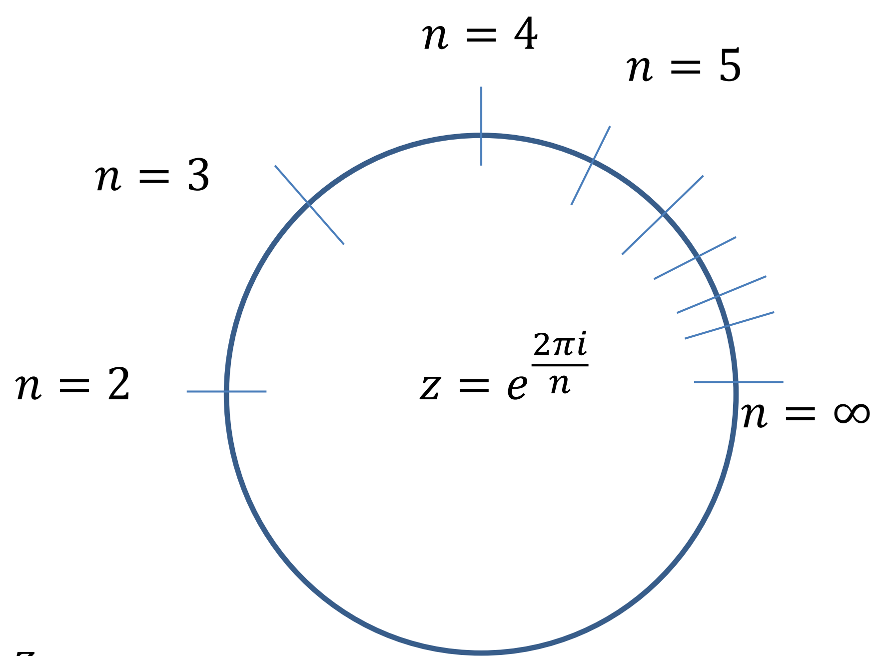
Discrete time systems

Continuous time

Discrete time



s plane



$$e^{sT} = z$$

(an eigenvalue s in the continuous-time system transforms into the eigenvalue $z = e^{sT}$ in a discrete-time system with sampling time T)

z plane

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Controllability

Definition: Given a system (A, B) of dimension n, a state w is reachable if there exists an input $u(k)$ of finite energy and a time m such that under that input and zero initial condition, the state of the system becomes w:

$$w(m) = \underbrace{A^m w_I}_{0} + \sum_{k=0}^{m-1} A^{m-k-1} B u(k)$$

Definition: the reachable subspace is the set containing all the reachable states of the system

Definition: $\mathcal{R}(A, B) = [B, AB, \cdots, A^k B, \cdots]$ is the reachability matrix.

Definition: $\mathcal{R}_m\,(A, B) = [B, AB, \cdots, A^{m-1} B\,]$

Theorem: the reachable subspace corresponds to im $\mathcal{R}(A, B)$ (or the span of the columns of $\mathcal{R}(A, B)$).

Definition: The system *(A,B)* is completely reachable if the reachable subspace is the full space, i.e. rank $\mathcal{R}(A, B) = n$.

Controllability

Definition:
The controllability Gramian at horizon m is the matrix:

$$G_c = \mathcal{R}_m\,(A,B)[\mathcal{R}_m\,(A,B)]^* = \sum_{k=0}^{m-1} A^k B B^* {A^*}^k$$

Properties of G_c:

1/ G_c is an n by n symmetric positive matrix: $w^* G_c w \geq 0\ \forall w$.

2/ $\mathrm{im}\, G_c = \mathrm{im}\, \mathcal{R}(A,B)$ for $m \geq n$

3/ (A,B) is fully reachable if and only if the kernel of G_c is the null space for some m

4/ If (A,B) is fully reachable, then the minimum energy to reach w over control horizon $[0,m]$ is $w^* G_c^{-1} w$. w is a controllable state if this quantity is small.

Controllability

Proof of $1/ G_c^* = G_c$

$$w^* G_c w = \sum_{k=0}^{m-1} w^* A^k B B^* {A^*}^k w = \sum_{k=0}^{m-1} \left| B^* {A^*}^k w \right|^2 \geq 0$$

Proof of $3/$ dim im G_c + dim Ker $G_c = n$

Controllability

Proof of 4/ for any state w_f, there exists a state ξ such that

$$w_f = G_c \xi.$$

Let us consider at time m the state obtained with the control law

$$u(k) = B^* {A^*}^{m-k-1} \xi$$

and the initial state $w(0) = w_I = 0$. Then:

$$w(m) = \underbrace{A^m w_I}_{0} + \sum_{k=0}^{m-1} A^{m-k-1} B u(k)$$

$$= \sum_{k=0}^{m-1} A^{m-k-1} B B^* {A^*}^{m-k-1} \xi \underbrace{=}_{k'=m-1-k} \underbrace{\sum_{k'=0}^{m-1} A^{k'} B B^* {A^*}^{k'}}_{G_c} \xi = w_f$$

The energy cost related to this law is:

$$\sum_{k=0}^{m-1} u(k)^* u(k) = \sum_{k=0}^{m-1} \xi^* A^{m-k-1} B B^* {A^*}^{m-k-1} \xi = \xi^* G_c \xi = \left(G_c^{-1} w_f\right)^* G_c G_c^{-1} w_f$$

$$= w_f^* G_c^{-1} w_f$$

Controllability

If $m \to \infty$, the infinite controllability Gramian is:

$$G_c^\infty = \sum_{k=0}^{\infty} A^k B B^* A^{*k}$$

G_c^∞ is solution of a discrete Lyapunov equation:

$$A G_c^\infty A^* - G_c^\infty + B B^* = 0$$

<u>Proof:</u>

$$\begin{aligned} G_c^\infty &= \sum_{k=0}^{\infty} A^k B B^* A^{*k} = B B^* + \sum_{k=1}^{\infty} A^k B B^* A^{*k} \\ &= B B^* + A \underbrace{\left(\sum_{k=1}^{\infty} A^{k-1} B B^* A^{*k-1} \right)}_{G_c^\infty} A^* \end{aligned}$$

So:

$$G_c^\infty = B B^* + A G_c^\infty A^*$$

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Observability

Definition: Given a system (A, C) of dimension n, a state w is unobservable if for all $k \geq 0, CA^k w = 0$.

Definition: the unobservable subspace is the set containing all the unobservable states of the system.

Definition: $\mathcal{O}(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^k \\ \vdots \end{bmatrix}$ is the observability matrix.

Definition: $\mathcal{O}_m(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix}$

Theorem: the unobservable subspace corresponds to $\ker \mathcal{O}(C, A)$

Observability

Definition: For a stable system, the observability Gramian at horizon m is

$$G_o = [\mathcal{O}_m(C,A)]^* \mathcal{O}_m(C,A) = \sum_{k=0}^{m-1} A^{*k} C^* C A^k$$

Properties:

1/ G_o is an n by n symmetric positive semi-definite matrix:

$$w^* G_o w = \sum_{k=0}^{m-1} \left| C A^k w \right|^2 \geq 0 \ \forall w$$

2/ $\ker \mathcal{O}(C,A)$=$\ker G_o$ for $m \geq n$ (the unobservable subspace is the kernel of the observability Gramian).

3/ The energy produced by observing the output of the system corresponding to an initial state w over time horizon $[0,m]$ is $w^* G_o w$. w is an observable state if this quantity is large.

Observability

The inifinite observability Gramian G_o^∞

$$G_o^\infty = \sum_{k=0}^{\infty} A^{*k} C^* C A^k$$

is solution of a discrete Lyapunov equation :

$$A^* \, G_o^\infty A - G_o^\infty + C^* C = 0$$

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Change of basis

If $w = Tw_r$, then

$$\begin{cases} w_r(k+1) = \overbrace{T^{-1}AT}^{A_r}\, w_r(k) + \overbrace{T^{-1}B}^{B_r}\, u(k) \\ y(k) = \underbrace{CT}_{C_r}\, w_r(k) \end{cases}$$

Then:

$$G_{cr} = \sum_{k=0}^{m-1} A_r^k B_r B_r^* A_r^{*k} = \sum_{k=0}^{m-1} T^{-1} A^k T T^{-1} B B^* T^{-1*} T^* A^{*k} T^{*-1}$$

$$= T^{-1} \underbrace{\sum_{k=0}^{m-1} A^k B B^* A^{*k}}_{G_c} T^{*-1}$$

$$G_{or} = \sum_{k=0}^{m-1} A_r^{*k} C_r^* C_r A_r^k = T^* \underbrace{\sum_{k=0}^{m-1} A^{*k} C^* C A^k}_{G_o} T$$

Change of basis

Theorem:
Under a change of basis, the product of the two Gramians becomes:

$$G_{cr}G_{or} = T^{-1}G_cG_oT.$$

The EVD of G_cG_o therefore yields a basis T in which $G_{cr}G_{or}$ is a diagonal matrix.

Proof:

$$G_{cr}G_{or} = T^{-1}G_cT^{*-1}\mathrm{T}^*\mathrm{G_o}\mathrm{T} = T^{-1}G_cG_oT$$

The EVD of $\mathrm{G_c}\mathrm{G_o}$ yields:

$$G_cG_o = T\Sigma T^{-1}$$

Then, in this basis:

$$G_{cr}G_{or} = T^{-1}T\Sigma T^{-1}\mathrm{T} = \Sigma$$

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Balanced representations

We look for states: That are easy to reach / That are easy to observe.
We would like to remove states: That are difficult to reach / That are difficult to observe.

Definition: A system (A, B, C) of dimension n is balanced if the controllability and observability Gramians are equal and diagonal:

$$G_c = G_o = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_n \end{pmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. These coefficients are the Hankel singular values. The vectors (e_i) of this basis verify:

$$e_i^* G_c^{-1} e_i = \frac{1}{\sigma_i},$$
$$e_i^* G_o e_i = \sigma_i.$$

Hence: e_1 has minimal reachability energy and maximal observability energy.
e_n has maximal reachability energy and minimal observability energy.

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How to obtain balanced representations?

Algorithm of Laub et al. 1987

1/ Factor G_c and G_o such that: $G_c = XX^*$ and $G_o = YY^*$

2/ Perform the SVD of the cross-Gramian:

$$Y^*X = U\Sigma V^*, V^*V = U^*U = I$$

The diagonal elements of Σ (Σ_j) are the Hankel singular values.

3/ Compute the bases: $T = XV\Sigma^{-\frac{1}{2}}$ and $S = YU\Sigma^{-\frac{1}{2}}$.

These bases are bi-orthogonal: $S^*T = I$

T provides a balanced system: $\mathrm{G}_{cr} = T^{-1}G_cT^{*-1} = \Sigma$, $G_{or} = T^*G_oT = \Sigma$

4/ $G_cG_o\mathrm{T} = \mathrm{T}\Sigma^2$

Proof:

$$S^*T = \Sigma^{-\frac{1}{2}}U^*Y^*XV\Sigma^{-\frac{1}{2}} = \Sigma^{-\frac{1}{2}}U^*U\Sigma V^*V\Sigma^{-\frac{1}{2}} = I$$

$$T^{-1}G_cT^{*-1} = \Sigma^{-\frac{1}{2}}U^*Y^*XX^*YU\Sigma^{-\frac{1}{2}} = \Sigma^{-\frac{1}{2}}U^*U\Sigma V^*V\Sigma U^*U\Sigma^{-\frac{1}{2}} = \Sigma$$

$$T^*G_oT = \Sigma^{-\frac{1}{2}}V^*X^*YY^*XV\Sigma^{-\frac{1}{2}} = \Sigma^{-\frac{1}{2}}V^*V\Sigma U^*U\Sigma V^*V\Sigma^{-\frac{1}{2}} = \Sigma$$

$$G_cG_o\mathrm{T} = XX^*YY^*XV\Sigma^{-\frac{1}{2}} = XV\Sigma U^*U\Sigma V^*V\Sigma^{-\frac{1}{2}} = \mathrm{T}\Sigma^2$$

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Balanced system/Balanced truncation

The balanced system is:

$$\begin{cases} w_r(k+1) = A_r w_r(k) + B_r u(k) \\ y(k) = C_r w_r(k) \end{cases}$$

with $A_r = S^* A T$, $B_r = S^* B$, $C_r = CT$

The balanced system may be rewritten as (the subscript 1 and 2 denoting respectively a subspace of dimension p and $n-p$):

$$\begin{bmatrix} w_{r1}(k+1) \\ w_{r2}(k+1) \end{bmatrix} = \begin{bmatrix} A_{r11} & A_{r12} \\ A_{r21} & A_{r22} \end{bmatrix} \begin{bmatrix} w_{r1}(k) \\ w_{r2}(k) \end{bmatrix} + \begin{bmatrix} B_{r1} \\ B_{r2} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} C_{r1} & C_{r2} \end{bmatrix} \begin{bmatrix} w_{r1}(k) \\ w_{r2}(k) \end{bmatrix}$$

The truncated system is:

$$w_{r1}(k+1) = A_{r11} w_{r1}(k) + B_{r1} u(k)$$

$$y(k) = C_{r1} w_{r1}(k)$$

The impulse response $Z_p(k) = C_{r1} A_{r11}^{k-1} B_{r1}$ and tranfer-function $T_{yu,p}(z) = C_{r1}(zI_r - A_{r11})^{-1} B_{r1}$ are close to the initial ones: $Z(k) = CA^{k-1}B$ and $T_{yu}(z) = C(zI - A)^{-1}B$.

Balanced truncation: error bounds

Error bounds:

With infinite Gramians, we have:

$$\Sigma_{p+1} < \left\| Z - Z_p \right\|_\infty < 2 \sum_{j=p+1}^{n} \Sigma_j$$

where Σ_j are the Hankel singular values.

Proof:

see Antoulas 2005.

Balanced truncation: stability

Stability:
Any reduced (truncated) system A_{r11} obtained by balanced truncation (with infinite Gramians) is stable.

Proof:
In the balanced basis, the infinite controllability Gramian verifies:

$$AG_c^\infty A^* - G_c^\infty + BB^* = 0$$

$$\Rightarrow \begin{bmatrix} A_{r11} & A_{r12} \\ A_{r21} & A_{r22} \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} A_{r11}^* & A_{r21}^* \\ A_{r12}^* & A_{r22}^* \end{bmatrix} - \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} + \begin{bmatrix} B_{r1} \\ B_{r2} \end{bmatrix} \begin{bmatrix} B_{r1}^* & B_{r2}^* \end{bmatrix} = 0$$

$$\begin{bmatrix} A_{r11} & A_{r12} \\ A_{r21} & A_{r22} \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} A_{r11}^* & A_{r21}^* \\ A_{r12}^* & A_{r22}^* \end{bmatrix} = \begin{bmatrix} A_{r11} & A_{r12} \\ A_{r21} & A_{r22} \end{bmatrix} \begin{bmatrix} \Sigma_1 A_{r11}^* & \Sigma_1 A_{r21}^* \\ \Sigma_2 A_{r12}^* & \Sigma_2 A_{r22}^* \end{bmatrix}$$
$$= \begin{bmatrix} A_{r11}\Sigma_1 A_{r11}^* + A_{r12}\Sigma_2 A_{r12}^* & ? \\ ? & ? \end{bmatrix}$$

Balanced truncation: stability

The upper left block is:

$$A_{r11}\Sigma_1 A_{r11}^* + A_{r12}\Sigma_2 A_{r12}^* - \Sigma_1 + B_{r1}B_{r1}^* = 0$$

Consider an eigenvalue/eigenvector:

$$A_{r11}^* w = \lambda w$$

Then:

$$\begin{aligned} w^* A_{r11}\Sigma_1 A_{r11}^* w + w^* A_{r12}\Sigma_2 A_{r12}^* w - w^*\Sigma_1 w + w^* B_{r1}B_{r1}^* w \\ = \underbrace{(|\lambda|^2 - 1)}_{=\sigma<0}\,\underbrace{w^*\Sigma_1 w}_{>0}\,\underbrace{+w^* A_{r12}\Sigma_2 A_{r12}^* w}_{>0}\,\underbrace{w^* B_{r1}B_{r1}^* w}_{>0} = 0 \end{aligned}$$

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- Observability
- Change of basis
- Balanced representations
- How to obtain balanced representations?
- Balanced truncation
  - Error bounds
  - Stability
- ERA

Eigensystem Realization Algorithm (ERA)

1/ Gramians may be factored as $G_c = XX^*$, $G_o = YY^*$ with:

$$X = \mathcal{R}_m\,(A, B) = [B, AB, \cdots, A^{m-1}B\,]$$

$$Y = [\mathcal{O}_m(C, A)]^* = [C^*, A^*C^*, \cdots, (A^*)^{m-1}C^*]$$

2/ $Y^*X = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} [B \quad AB \quad A^2B \quad \cdots] = \begin{bmatrix} CB & CAB & \cdots \\ CAB & CA^2B & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} Z(1) & Z(2) & \cdots \\ Z(2) & Z(3) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$

Can be obtained just from impulse response $Z(k) = CA^{k-1}B$.

2/Perform SVD of Y^*X:

$$Y^*X = U\Sigma V^* \approx U_r\Sigma_r V_r^*, U_r^*U_r = V_r^*V_r = I$$

3/ Compute bases: $T = XV_r\Sigma_r^{-\frac{1}{2}}$ and $S = YU_r\Sigma_r^{-\frac{1}{2}}$

4/ The balanced system is:

$$\begin{cases} w_r(k+1) = A_r w_r(k) + B_r u(k) \\ \qquad\quad y(k) = C_r w_r(k) \end{cases}$$

With $A_r = S^*AT$, $B_r = S^*B$, $C_r = CT$

Can we obtain (A_r, B_r, C_r) just from impulse response $Z(k)$?

Eigensystem Realization Algorithm (ERA)

$$1/\ A_r = S^*AT = \Sigma_r^{-\frac{1}{2}} U_r^* Y^* A X V_r \Sigma_r^{-\frac{1}{2}} = \Sigma_r^{-\frac{1}{2}} U_r^* \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} A \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix} V_r \Sigma_r^{-\frac{1}{2}} =$$

$$\Sigma_r^{-\frac{1}{2}} U_r^* \begin{bmatrix} CAB & CA^2B & \cdots \\ CA^2B & CA^3B & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} V_r \Sigma_r^{-\frac{1}{2}} = \Sigma_r^{-\frac{1}{2}} U_r^* \begin{bmatrix} Z(2) & Z(3) & \cdots \\ Z(3) & Z(4) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} V_r \Sigma_r^{-\frac{1}{2}}$$

$$2/\ B_r = S^*B = \Sigma_r^{-\frac{1}{2}} U_r^* Y^* \mathrm{B} = \Sigma_r^{-\frac{1}{2}} U_r^* \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} \mathrm{B} = \Sigma_r^{-\frac{1}{2}} U_r^* \begin{bmatrix} CB \\ CAB \\ CA^2B \\ \vdots \end{bmatrix}$$

Since $\Sigma_r^{-\frac{1}{2}} U_r^* (Y^*X) = \Sigma_r^{\frac{1}{2}} V_r^*$,

B_r corresponds to the first column of $\Sigma_r^{\frac{1}{2}} V_r^*$.

$$3/\ C_r = CT = CXV_r\Sigma_r^{-\frac{1}{2}} = \begin{bmatrix} CB & CAB & CA^2B & \cdots \end{bmatrix} V_r \Sigma_r^{-\frac{1}{2}}$$

Since $(Y^*X) V_r \Sigma_r^{-\frac{1}{2}} = U_r \Sigma_r^{\frac{1}{2}}$.

C_r corresponds to the first row of $U_r \Sigma_r^{\frac{1}{2}}$

Eigensystem Realization Algorithm (ERA)

<u>ERA algorithm:</u>
Let $Z(k) = CA^{k-1}B$ be the impulse response of a large-scale system.
If we perform the SVD of the Hankel matrix:

$$\begin{bmatrix} Z(1) & Z(2) & \cdots \\ Z(2) & Z(3) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} = U\Sigma V^* \approx U_r\Sigma_r V_r^*, U_r^* U_r = V_r^* V_r = I$$

then, the balanced system is:

$$\begin{cases} w_r(k+1) = A_r w_r(k) + B_r u(k) \\ y(k) = C_r w_r(k) \end{cases}$$

where:

1/ $A_r = \Sigma_r^{-\frac{1}{2}} U_r^* \begin{bmatrix} Z(2) & Z(3) & \cdots \\ Z(3) & Z(4) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} V_r \Sigma_r^{-\frac{1}{2}}$

2/ B_r corresponds to the first column of $\Sigma_r^{\frac{1}{2}} V_r^*$

3/ C_r corresponds to the first row of $U_r \Sigma_r^{\frac{1}{2}}$