

# Adjoints

# Outline

- Governing equations
- Asymptotic development
  - Order  $\epsilon^0$  : Base-flow
  - Order  $\epsilon^1$  : Global modes
- Bi-orthogonal basis and adjoint global modes
  - Definition of adjoint global modes
  - Optimal initial condition
  - Optimal forcing in stable flow
- Adjoint operator
  - Definition
  - Adjoint global modes as solutions of adjoint eigen-problem
- Adjoint linearized Navier-Stokes operator
  - Adjoint of linearized advection operator
  - Adjoint of Stokes operator
  - Adjoint global modes of cylinder flow

# Governing equations

Incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t u + u \partial_x u + v \partial_y u = -\partial_x p + \nu(\partial_{xx} u + \partial_{yy} u) + f \\ \partial_t v + u \partial_x v + v \partial_y v = -\partial_y p + \nu(\partial_{xx} v + \partial_{yy} v) + g \\ -\partial_x u - \partial_y v = 0 \end{cases}$$

Can be recast into:

$$\mathcal{B} \partial_t w + \frac{1}{2} \mathcal{N}(w, w) + \mathcal{L} w = f$$

where:

$$\begin{aligned} w &= \begin{pmatrix} u \\ p \end{pmatrix} \quad f = \begin{pmatrix} f \\ 0 \end{pmatrix} \\ \mathcal{B} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ \mathcal{N}(w_1, w_2) &= \begin{pmatrix} u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 \\ 0 \end{pmatrix} \\ \mathcal{L} &= \begin{pmatrix} -\nu \Delta \cdot \cdot & \nabla \cdot \cdot \\ -\nabla \cdot \cdot & 0 \end{pmatrix} \end{aligned}$$

Boundary conditions: Dirichlet, Neumann, Mixed

# Some properties

$$a) \mathcal{N}(w_1, w_2) = \mathcal{N}(w_2, w_1)$$

$$b) \frac{1}{2} \mathcal{N}(w_0 + \epsilon \delta w, w_0 + \epsilon \delta w) = \frac{1}{2} \mathcal{N}(w_0, w_0) + \underbrace{\epsilon \mathcal{N}(w_0, \delta w)}_{\text{Jacobian}} + \frac{\epsilon^2}{2} \underbrace{\mathcal{N}(\delta w, \delta w)}_{\text{Hessian}} + \dots \\ = \mathcal{N}_{w_0} \delta w$$

$$c) \mathcal{N}_{w_0} \delta w = \mathcal{N}(w_0, \delta w) = \begin{pmatrix} \delta u \cdot \nabla u_0 + u_0 \cdot \nabla \delta u \\ 0 \end{pmatrix}$$

$$d) \mathcal{B}w = \mathcal{B} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix}$$

e)  $\partial_t u + u \cdot \nabla u = -\nabla p + \nu \nabla^2 u \Rightarrow -\nabla^2 p = \nabla \cdot (u \cdot \nabla u)$ ,  $\partial_n p = \nu \nabla^2 u \cdot n$  on solid walls. Hence,  $p$  is a function of  $u$  and should not be considered as a degree of freedom of the flow.

f) Scalar-product:  $\langle w_1, w_2 \rangle = \iint (u_1^* u_2 + v_1^* v_2) dx dy = \iint (w_1 \cdot \mathcal{B}w_2) dx dy$  so that  $\sqrt{\{\langle w, w \rangle\}}$  is the energy.

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# Asymptotic development

Solution:

$$w(t) = w_0 + \epsilon w_1(t) + \dots \text{ with } \epsilon \ll 1$$

Governing equations:

$$\mathcal{B}\partial_t w + \frac{1}{2} \mathcal{N}(w, w) + \mathcal{L}w = f$$

Introduce solution into governing eq::

$$\mathcal{B}\partial_t(w_0 + \epsilon w_1 + \dots) + \frac{1}{2} \mathcal{N}(w_0 + \epsilon w_1 + \dots, w_0 + \epsilon w_1 + \dots) + \mathcal{L}(w_0 + \epsilon w_1 + \dots) = f$$

$$\Rightarrow \begin{cases} \frac{1}{2} \mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = f \text{ at order } O(1) \\ \mathcal{B}\partial_t w_1 + \underbrace{\frac{1}{2} [\mathcal{N}(w_1, w_0) + \mathcal{N}(w_0, w_1)]}_{\mathcal{N}_{w_0} w_1} + \mathcal{L}w_1 = 0 \text{ at order } O(\epsilon) \\ \mathcal{B}\partial_t w_2 + \mathcal{N}_{w_0} w_2 + \mathcal{L}w_2 = -\frac{1}{2} \mathcal{N}(w_1, w_1) \text{ at order } O(\epsilon^2) \end{cases}$$

# Oder $\epsilon^0$ : Base-flow

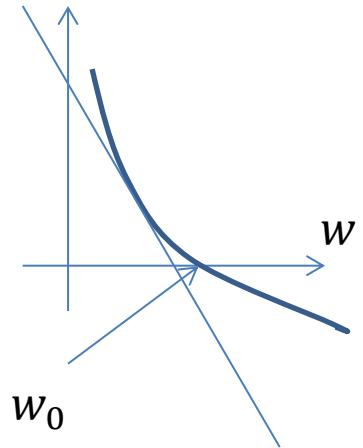
Definition:

$$w(t) = w_0 + \epsilon w_1(t) + \dots$$

Non-linear equilibrium point :

$$\frac{1}{2} \mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = f$$

$$F(w) = \frac{1}{2} \mathcal{N}(w, w) + \mathcal{L}w - f$$



How to compute a base-flow ?

Newton iteration:

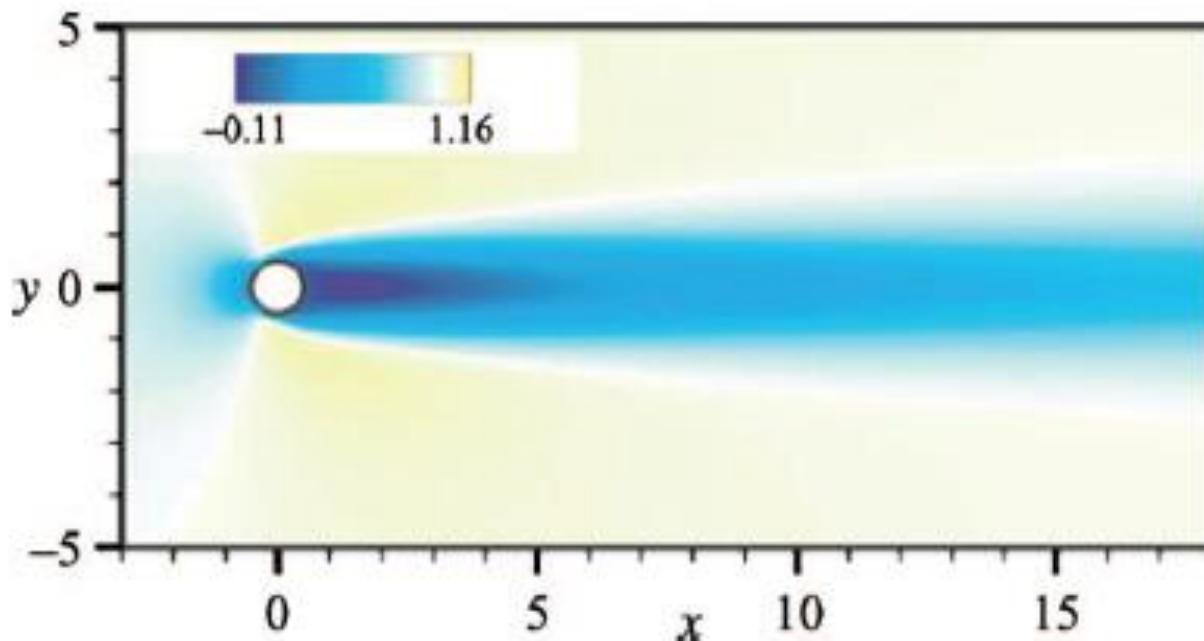
$$\frac{1}{2} \mathcal{N}(w_0 + \delta w_0, w_0 + \delta w_0) + \mathcal{L}(w_0 + \delta w_0) = f$$

Linearization:

$$\begin{aligned} \mathcal{N}(w_0, \delta w_0) + \mathcal{L}\delta w_0 &= f - \frac{1}{2} \mathcal{N}(w_0, w_0) - \mathcal{L}w_0 \\ \Rightarrow \delta w_0 &= (\mathcal{N}_{w_0} + \mathcal{L})^{-1} \left( f - \frac{1}{2} \mathcal{N}(w_0, w_0) - \mathcal{L}w_0 \right) \end{aligned}$$

# Oder $\epsilon^0$ : Base-flow

## The case of cylinder flow



$Re = 47$   
Streamwise velocity field of base-flow.

# Order $\epsilon^1$ : Global modes

## Definition

$$w(t) = w_0 + \epsilon w_1(t) + \dots$$

Linear governing equation:

$$\mathcal{B}\partial_t w_1 + \mathcal{N}_{w_0} w_1 + \mathcal{L}w_1 = 0$$

Solution  $w_1$  under the form:

$$w_1 = e^{\lambda t} \hat{w} + \text{c.c}$$

This leads to :

$$\lambda \mathcal{B} \hat{w} + (\mathcal{N}_{w_0} + \mathcal{L}) \hat{w} = 0$$

Eigenvalue:

$$\lambda = \sigma + i\omega$$

Eigenvector:

$$\hat{w} = \hat{w}_r + i\hat{w}_i$$

Real solution:

$$w_1 = e^{\lambda t} \hat{w} + \text{c.c} = 2e^{\sigma t} (\cos \omega t \hat{w}_r - \sin \omega t \hat{w}_i)$$

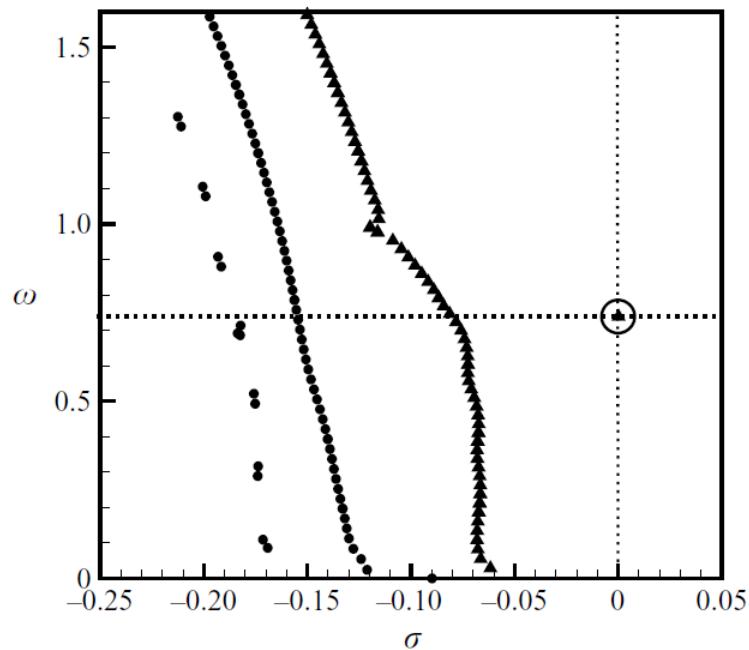
# Order $\epsilon^1$ : Global modes

## How to compute global modes ?

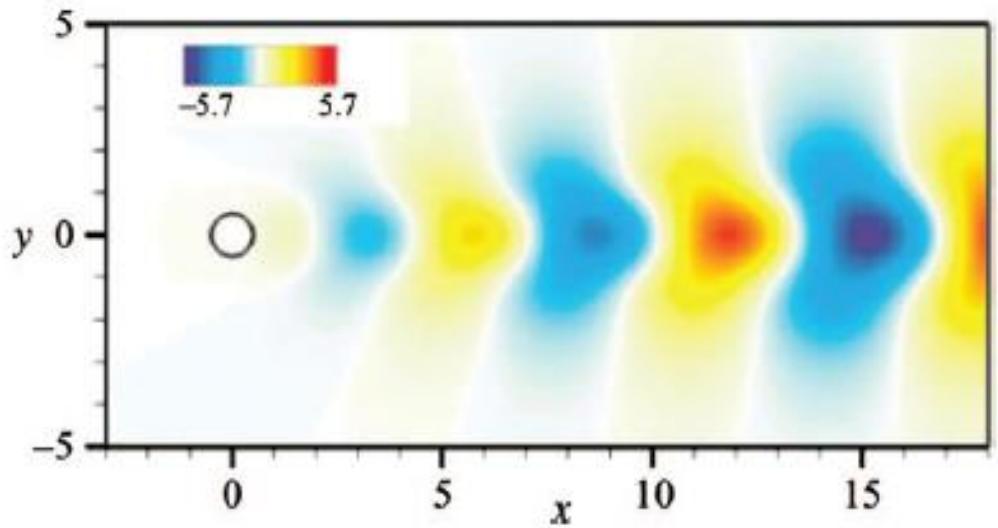
Eigenvalue problem solved with shift-invert strategy:

- Power method, easy to find largest magnitude eigenvalues of  $Ax = \lambda x$ . For this, evaluate  $A^n x_0$
- To find eigenvalues of  $A$  closest to zero, search largest magnitude eigenvalues of  $A^{-1}$ :  $A^{-1}x = \lambda^{-1}x$ . For this, evaluate  $(A^{-1})^n x_0$
- To find eigenvalues of  $A$  closest to  $s$ , search largest magnitude eigenvalues of  $(A - sI)^{-1}$ :  $(A - sI)^{-1}x = (\lambda - s)^{-1}x$ . For this, evaluate  $((A - sI)^{-1})^n x_0$
- Instead of power-method, use Krylov subspaces -> Arnoldi technique
- Cost of algorithm = cost of several complex matrix inversions

# Order $\epsilon^1$ : Global modes Case of cylinder flow



Spectrum  $Re = 47$



Real part of cross-stream velocity field  
Marginal eigenmode

# The Ginzburg-Landau eq.

We consider the linear Ginzburg-Landau equation

$$\partial_t w_1 + \mathcal{L} w_1 = 0$$

where

$$\mathcal{L} = U\partial_x - \mu(x) - \gamma\partial_{xx}, \quad \mu(x) = i\omega_0 + \mu_0 - \mu_2 \frac{x^2}{2}.$$

Here  $U, \gamma, \omega_0, \mu_0$  and  $\mu_2$  are positive real constants. The state  $w(x, t)$  is a complex variable on  $-\infty < x < +\infty$  such that  $|w| \rightarrow 0$  as  $|x| \rightarrow \infty$ . In the following,

$$\langle w_a, w_b \rangle = \int_{-\infty}^{+\infty} w_a(x)^* w_b(x) dx .$$

1/ What do the different terms in the Ginzburg Landau equation represent?

# The Ginzburg-Landau eq.

2/ Show that  $\hat{w}(x) = \zeta e^{\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$  with  $\chi = \left(\frac{\mu_2}{2\gamma}\right)^{\frac{1}{4}}$  and  $\zeta = \frac{\sqrt{\chi}}{\pi^{\frac{1}{4}} e^{\frac{1}{8\gamma^2 \chi^2}}}$  verifies  $\lambda \hat{w} + \mathcal{L} \hat{w} = 0$ .

What is the eigenvalue  $\lambda$  associated to this eigenvector? The constant  $\zeta$  has been selected so that  $\langle \hat{w}, \hat{w} \rangle = 1$ .

3/ Show that the flow is unstable if the constant  $\mu_0$  is chosen such that:  $\mu_0 > \mu_c$ , where  $\mu_c = \frac{U^2}{4\gamma} + \sqrt{\frac{\gamma\mu_2}{2}}$ .

Nota:  $\left( \lambda_n = i\omega_0 + \mu_0 - \frac{U^2}{4\gamma} - (2n+1)\sqrt{\frac{\gamma\mu_2}{2}}, \hat{w}_n = \zeta_n H_n(\chi x) e^{\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}} \right)$  are all the eigenvalues/eigenvectors of  $\mathcal{L}$ ,  $H_n$  being Hermite polynomials.

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# Bi-orthogonal basis and adjoint global modes (1/3)

## In finite dimension

**Global modes:**

$$A\hat{w}_i = \lambda_i \hat{w}_i$$

The eigenvectors  $\hat{w}_i$  form a basis:

$$w = \sum_i \alpha_i \hat{w}_i$$

**Definition of adjoint global modes:** with  $\langle \cdot, \cdot \rangle$  as a given scalar-product (say  $\langle w_1, w_2 \rangle = w_1^* w_2$ ), there exists for each  $\alpha_i$  a unique  $\tilde{w}_i$  such that  $\alpha_i = \langle \tilde{w}_i, w \rangle$  for all  $w$ . The adjoint global modes are the structures  $\tilde{w}_i$ . In the following:  $\langle \hat{w}_i, \hat{w}_i \rangle = 1$ .

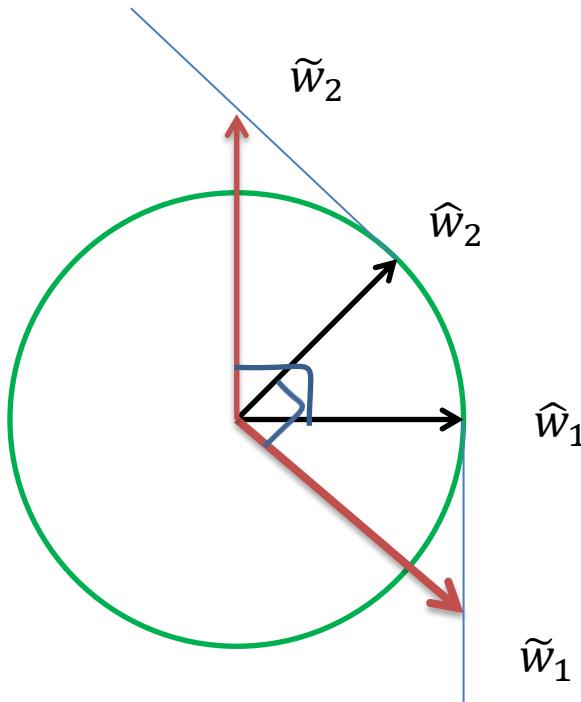
**Properties:**

- $\tilde{w}_k$  and  $\hat{w}_j$  are bi-orthogonal bases: they verify  $\hat{w}_j = \sum_i \langle \tilde{w}_i, \hat{w}_j \rangle \hat{w}_i$  and so  $\langle \tilde{w}_k, \hat{w}_j \rangle = \delta_{kj}$  (in matrix notations  $\tilde{W}^* \hat{W} = I$ )
- Cauchy-Lipschitz:  $1 = |\langle \tilde{w}_i, \hat{w}_i \rangle| \leq \langle \tilde{w}_i, \tilde{w}_i \rangle^{\frac{1}{2}} \langle \hat{w}_i, \hat{w}_i \rangle^{\frac{1}{2}}$   
Hence:  $\langle \tilde{w}_i, \tilde{w}_i \rangle^{\frac{1}{2}} \geq 1$  and  $\cos \text{angle}(\tilde{w}_i, \hat{w}_i) = \frac{1}{\langle \tilde{w}_i, \tilde{w}_i \rangle^{\frac{1}{2}}}$

# Bi-orthogonal basis and adjoint global modes (2/3)

In finite dimension

$$w = (\tilde{w}_1 \cdot w) \hat{w}_1 + (\tilde{w}_2 \cdot w) \hat{w}_2$$



Def of  $\tilde{w}_1$ :

$$\begin{aligned}\tilde{w}_1 \cdot \hat{w}_1 &= 1 \\ \tilde{w}_1 \cdot \hat{w}_2 &= 0\end{aligned}$$

Def of  $\tilde{w}_2$ :

$$\begin{aligned}\tilde{w}_2 \cdot \hat{w}_2 &= 1 \\ \tilde{w}_2 \cdot \hat{w}_1 &= 0\end{aligned}$$

$$\tilde{W}^* \hat{W} = I$$

Method 1 :  $\tilde{W} = \hat{W}^{*-1}$

Method 2 :  $\tilde{W} = \hat{W}X \Rightarrow X^* \hat{W}^* \hat{W} = I \Rightarrow X = (\hat{W}^* \hat{W})^{-1} \Rightarrow \tilde{W} = \hat{W}(\hat{W}^* \hat{W})^{-1}$

Method 3 : adjoint global modes

# Bi-orthogonal basis and adjoint global modes (3/3)

Global modes:

$$\lambda_i \mathcal{B} \hat{w}_i + (\mathcal{N}_{w_0} + \mathcal{L}) \hat{w}_i = 0$$

The eigenvectors  $\hat{w}_i$  form a basis:

$$w = \sum_i \alpha_i \hat{w}_i$$

**Definition of adjoint global modes:** with  $\langle \cdot, \cdot \rangle$  as a given scalar-product, there exists for each  $\alpha_i$  a unique  $\tilde{w}_i$  such that  $\alpha_i = \langle \tilde{w}_i, \mathcal{B}w \rangle$  for all  $w$ . The adjoint global modes are the structures  $\tilde{w}_i$ . In the following:  $\langle \hat{w}_i, \mathcal{B} \hat{w}_i \rangle = 1$ .

Properties:

- $\tilde{w}_k$  and  $\hat{w}_j$  are bi-orthogonal bases: they verify  $\hat{w}_j = \sum_i \langle \tilde{w}_i, \mathcal{B} \hat{w}_j \rangle \hat{w}_i$  and so  $\langle \tilde{w}_k, \mathcal{B} \hat{w}_j \rangle = \delta_{kj}$
- Cauchy-Lipschitz:  $1 = |\langle \tilde{w}_i, \mathcal{B} \hat{w}_i \rangle| \leq \langle \tilde{w}_i, \mathcal{B} \tilde{w}_i \rangle^{\frac{1}{2}} \langle \hat{w}_i, \mathcal{B} \hat{w}_i \rangle^{\frac{1}{2}}$ . Hence:  
 $\langle \tilde{w}_i, \mathcal{B} \tilde{w}_i \rangle^{\frac{1}{2}} \geq 1$  and  $\cos \text{angle}(\tilde{w}_i, \hat{w}_i) = \frac{1}{\sqrt{\langle \tilde{w}_i, \mathcal{B} \tilde{w}_i \rangle}}$

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# Optimal initial condition (1/3)

## Definition of optimal initial condition

Initial-value problem:

$$\mathcal{B}\partial_t w_1 + (\mathcal{N}_{w_0} + \mathcal{L})w_1 = 0, \quad w_1(t=0) = w^I$$

Solution:

$$w_1(t) = \sum_i \langle \tilde{w}_i, \mathcal{B}w^I \rangle e^{\lambda_i t} \hat{w}_i$$

If  $(\hat{w}_1, \lambda_1)$  is the global mode which displays largest growth rate, at large times:

$$w_1(t) \approx \langle \tilde{w}_1, \mathcal{B}w^I \rangle e^{\lambda_1 t} \hat{w}_1$$

We look for unit-norm  $w^I$  ( $\langle w^I, \mathcal{B}w^I \rangle = 1$ ) which maximizes the amplitude of the response at large times.  $w_I$  is the optimal initial condition.

# Optimal initial condition (2/3)

If direct global mode as initial condition:

$$w^I = \hat{w}_1$$

In this case, at large time:

$$w_1(t) \approx e^{\lambda_1 t} \hat{w}_1$$

If adjoint global mode as initial condition:

$$w^I = \frac{\tilde{w}_1}{\sqrt{\langle \tilde{w}_1, \mathcal{B}\tilde{w}_1 \rangle}}$$

Then, at large time:

$$w_1(t) \approx \sqrt{\langle \tilde{w}_1, \mathcal{B}\tilde{w}_1 \rangle} e^{\lambda_1 t} \hat{w}_1$$

This is optimal since:

$$|\langle \tilde{w}_1, \mathcal{B}w^I \rangle| \leq \sqrt{\langle \tilde{w}_1, \mathcal{B}\tilde{w}_1 \rangle} \underbrace{\sqrt{\langle w^I, \mathcal{B}w^I \rangle}}_1$$

# Optimal initial condition (3/3)

Estimation of gain:

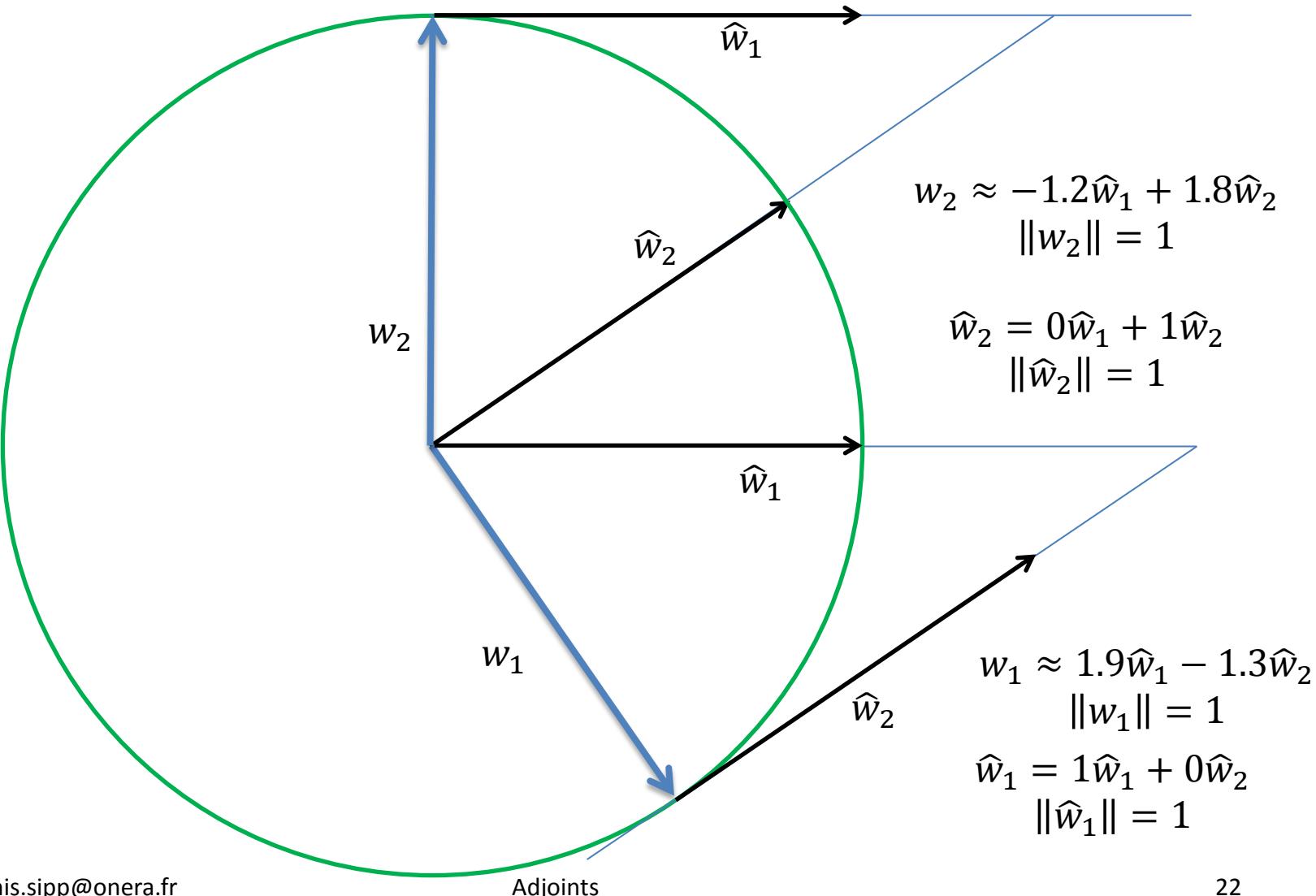
From Causchy-Lipschitz:

$$\langle \tilde{w}_1, \mathcal{B}\tilde{w}_1 \rangle^{\frac{1}{2}} \geq 1$$

Amplitude gain:

$$\langle \tilde{w}_1, \mathcal{B}\tilde{w}_1 \rangle^{\frac{1}{2}} = \frac{1}{\cos \text{angle}(\hat{w}_1, \tilde{w}_1)}$$

# In finite dimension



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# Optimal forcing in stable flow (1/2)

Problem:

$$\begin{aligned}\mathcal{B}\partial_t w + \frac{1}{2}\mathcal{N}(w, w) + \mathcal{L}w &= \epsilon\mathcal{B}f_1 \\ w &= w_0 + \epsilon w_1\end{aligned}$$

At first order:  $\mathcal{B}\partial_t w_1 + (\mathcal{N}_{w_0} + \mathcal{L})w_1 = f_1$

In frequency domain:  $w_1 = e^{i\omega t}\hat{w}$  and  $f_1 = e^{i\omega t}\hat{f}$

Governing equation:

$$i\omega\mathcal{B}\hat{w} + (\mathcal{N}_{w_0} + \mathcal{L})\hat{w} = \mathcal{B}\hat{f}$$

Where to force ( $\hat{f}$ ) and at which frequency ( $\omega$ ) to obtain strongest response ( $\hat{w}$ )?

# Optimal forcing in stable flow (1/2)

Introducing global mode basis:  $\widehat{w} = \sum_i \langle \tilde{w}_i, \mathcal{B}\widehat{w} \rangle \widehat{w}_i$  and  $\mathcal{B}\widehat{f} = \sum_i \langle \tilde{w}_i, \mathcal{B}\widehat{f} \rangle \mathcal{B}\widehat{w}_i$ :

$$\sum_i i\omega \langle \tilde{w}_i, \mathcal{B}\widehat{w} \rangle \mathcal{B}\widehat{w}_i - \lambda_i \langle \tilde{w}_i, \mathcal{B}\widehat{w} \rangle \mathcal{B}\widehat{w}_i = \sum_i \langle \tilde{w}_i, \mathcal{B}\widehat{f} \rangle \mathcal{B}\widehat{w}_i$$

Scalar-product with  $\tilde{w}_j$  and using bi-orthogonality:  $\langle \tilde{w}_j, \mathcal{B}\widehat{w}_i \rangle = \delta_{ij}$

$$\begin{aligned} \langle \tilde{w}_j, \mathcal{B}\widehat{w} \rangle (i\omega - \lambda_j) &= \langle \tilde{w}_j, \mathcal{B}\widehat{f} \rangle \\ \langle \tilde{w}_j, \mathcal{B}\widehat{w} \rangle &= \frac{\langle \tilde{w}_j, \mathcal{B}\widehat{f} \rangle}{i\omega - \lambda_j} \end{aligned}$$

Solution:

$$w_1(t) = e^{i\omega t} \sum_i \frac{\langle \tilde{w}_i, \mathcal{B}\widehat{f} \rangle}{i\omega - \lambda_i} \mathcal{B}\widehat{w}_i$$

To maximize response:

a/ force at frequencies  $i\omega$  closest to  $\lambda_i$

b/ force with  $\widehat{f} = \frac{\tilde{w}_i}{\langle \tilde{w}_i, \mathcal{B}\tilde{w}_i \rangle^{\frac{1}{2}}}$

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# Adjoint operator

## Definition

### Definition of adjoint operator:

Let  $\langle w_1, w_2 \rangle$  be a scalar product and  $\mathcal{A}$  a linear operator.

The adjoint operator of  $\mathcal{A}$  verifies  $\langle w_1, \mathcal{A}w_2 \rangle = \langle \tilde{\mathcal{A}}w_1, w_2 \rangle$  whatever  $w_1$  and  $w_2$ .

# Adjoint operator

## Example in finite dimension

Space:

$$w \in \mathbb{C}^N$$

Scalar-product:

$$\langle w_1, w_2 \rangle = w_1^* Q w_2$$

with  $Q$  a Hermitian matrix  $Q^* = Q$ .

Linear operator:  $\mathcal{A}$  matrix.

Adjoint operator:

$$\langle w_1, \mathcal{A}w_2 \rangle = w_1^* Q \mathcal{A} w_2 = w_1^* Q \mathcal{A} Q^{-1} Q w_2 = (Q^{-1} \mathcal{A}^* Q w_1)^* Q w_2 = \langle \tilde{\mathcal{A}} w_1, w_2 \rangle$$

with  $\tilde{\mathcal{A}} = Q^{-1} \mathcal{A}^* Q$

If  $Q = I$ , then  $\tilde{\mathcal{A}} = \mathcal{A}^*$

# The Ginzburg-Landau eq. (cont'd)

4/ Determine the operator  $\tilde{\mathcal{L}}$  adjoint to  $\mathcal{L}$ , considering the scalar product  $\langle \cdot, \cdot \rangle$ .

# Adjoint operator

## Example with linear PDE and B.C. (1/2)

Space:

Functions  $x \in [0,1] \rightarrow \mathbb{C}$  such that  $u(0) = \partial_x u(1) = 0$ .

Scalar-product:

$$\langle u_1, u_2 \rangle = \int_0^1 u_1^* u_2 dx$$

Linear operator  $\mathcal{A}$ :

$$\mathcal{A}u = U\partial_x u - \alpha u - \nu\partial_{xx} u$$

Adjoint operator:

$$\begin{aligned}\langle u_1, \mathcal{A}u_2 \rangle &= \int_0^1 u_1^*(U\partial_x u_2 - \alpha u_2 - \nu\partial_{xx} u_2) dx = \\ \int_0^1 (u_1^* U \partial_x u_2 - \alpha u_1^* u_2 - \nu u_1^* \partial_{xx} u_2) dx &= \\ [u_1^* U u_2 - \nu u_1^* \partial_x u_2]_0^1 + \int_0^1 (-\partial_x(u_1^* U) u_2 - \alpha u_1^* u_2 + \nu \partial_x u_1^* \partial_x u_2) dx &= \\ [u_1^* U u_2 - \nu u_1^* \partial_x u_2 + \nu (\partial_x u_1^*) u_2]_0^1 + \int_0^1 (-\partial_x(U u_1) - \alpha u_1 - \nu \partial_{xx} u_1)^* u_2 dx &= \\ \langle \tilde{\mathcal{A}}u_1, u_2 \rangle &\end{aligned}$$

Hence:

$$\tilde{\mathcal{A}}u = -\partial_x(Uu) - \alpha u - \nu \partial_{xx} u = -U\partial_x u - u\partial_x U - \alpha u - \nu \partial_{xx} u$$

# Adjoint operator

## Example with linear PDE and B.C. (2/2)

Boundary integral term:  $[u_1^* U u_2 - \nu u_1^* \partial_x u_2 + \nu (\partial_x u_1^*) u_2]_0^1 = 0$

At  $x = 0$ :  $u_2 = 0$  and  $\partial_x u_2 \neq 0$ , so that  $u_1 = 0$

At  $x = 1$ :  $\partial_x u_2 = 0$  and  $u_2 \neq 0$ , so that  $u_1^* U + \nu (\partial_x u_1^*) = 0$ , or  $u_1 U + \nu \partial_x u_1 = 0$

$u_1$  should be in the following space:

Functions  $x \in [0,1] \rightarrow \mathbb{C}$  such that  $u(0) = u_1(1)U + \nu \partial_x u(1) = 0$ .

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# Adjoint global modes and biorthogonality (1/4)

## In finite dimension

Theorem:

Let  $(\hat{w}_i, \lambda_i)$  be eigenvalues/eigenvectors of  $A\hat{w}_i = \lambda_i \hat{w}_i$ . Then there exists  $(\tilde{w}_i, \lambda_i^*)$  solution of the adjoint eigenproblem  $A^*\tilde{w}_i = \lambda_i^* \tilde{w}_i$ . These structures are the adjoint global modes and may be scaled such that  $\tilde{w}_i^* \hat{w}_j = \delta_{ij}$ . The vectors  $\tilde{w}_i$  are bi-orthogonal with respect to the vectors  $\hat{w}_j$ .

# Adjoint global modes and biorthogonality (2/4)

In finite dimension

Proof:

$$\lambda_i \hat{w}_i = A \hat{w}_i$$

$$\lambda_j^* \tilde{w}_j = A^* \tilde{w}_j$$

$$\begin{aligned} \lambda_i \tilde{w}_j^* \hat{w}_i &= \tilde{w}_j^* A \hat{w}_i = (A^* \tilde{w}_j)^* \hat{w}_i = (\lambda_j^* \tilde{w}_j)^* \hat{w}_i = \lambda_j \tilde{w}_j^* \hat{w}_i \\ &\quad (\lambda_i - \lambda_j) \tilde{w}_j^* \hat{w}_i = 0 \end{aligned}$$

If  $\lambda_i \neq \lambda_j$ , then  $\tilde{w}_j^* \hat{w}_i = 0$

If  $\tilde{w}_j^* \hat{w}_i \neq 0$ , then  $\lambda_i = \lambda_j$ .

Conclusion:  $\tilde{w}_j$  can be chosen such that

$$\tilde{w}_j^* \hat{w}_i = \delta_{ji}$$

# Adjoint global modes and biorthogonality (3/4)

Theorem:

Let  $(\hat{w}_i, \lambda_i)$  be eigenvalues/eigenvectors of  $\lambda_i \mathcal{B} \hat{w}_i + (\mathcal{N}_{w_0} + \mathcal{L}) \hat{w}_i = 0$ . Then there exists  $(\tilde{w}_i, \lambda_i^*)$  solution of the adjoint eigenproblem  $\lambda_i^* \mathcal{B} \tilde{w}_i + (\tilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}}) \tilde{w}_i = 0$ . These structures are the adjoint global modes and may be scaled such that  $\langle \tilde{w}_i, \mathcal{B} \hat{w}_j \rangle = \delta_{ij}$ . The vectors  $\tilde{w}_i$  are bi-orthogonal with respect to the vectors  $\hat{w}_j$ .

# Adjoint global modes and biorthogonality (4/4)

Proof:

$$\begin{aligned}\lambda_i \mathcal{B} \hat{w}_i + (\mathcal{N}_{w_0} + \mathcal{L}) \hat{w}_i &= 0 \\ \lambda_j^* \mathcal{B} \tilde{w}_j + (\tilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}}) \tilde{w}_j &= 0 \\ \langle \tilde{w}_j, (\mathcal{N}_{w_0} + \mathcal{L}) \hat{w}_i \rangle &= -\lambda_i \langle \tilde{w}_j, \mathcal{B} \hat{w}_i \rangle \\ \langle \tilde{w}_j, (\mathcal{N}_{w_0} + \mathcal{L}) \hat{w}_i \rangle &= \langle (\tilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}}) \tilde{w}_j, \hat{w}_i \rangle = \langle -\lambda_j^* \mathcal{B} \tilde{w}_j, \hat{w}_i \rangle = -\lambda_j \langle \tilde{w}_j, \mathcal{B} \hat{w}_i \rangle \\ (\lambda_i - \lambda_j) \langle \tilde{w}_j, \mathcal{B} \hat{w}_i \rangle &= 0\end{aligned}$$

If  $\lambda_i \neq \lambda_j$ , then  $\langle \tilde{w}_j, \mathcal{B} \hat{w}_i \rangle = 0$

If  $\langle \tilde{w}_j, \mathcal{B} \hat{w}_i \rangle \neq 0$ , then  $\lambda_i = \lambda_j$ .

Conclusion:  $\tilde{w}_j$  can be chosen such that

$$\langle \tilde{w}_j, \mathcal{B} \hat{w}_i \rangle = \delta_{ji}$$

# The Ginzburg-Landau eq. (cont'd)

5/ Show that:  $\tilde{w}(x) = \xi e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$  with  $\xi = \sqrt{\chi}\pi^{-\frac{1}{4}}$  is solution of  $\lambda^* \tilde{w} + \tilde{\mathcal{L}} \tilde{w} = 0$ . Note that the normalization constant  $\xi$  has been chosen so that:

$$\langle \tilde{w}, \hat{w} \rangle = 1.$$

Can you qualitatively represent  $\hat{w}(x)$  and  $\tilde{w}(x)$ ?

6/ Noting that:

$$\langle \tilde{w}, \tilde{w} \rangle = e^{\frac{1}{2\sqrt{2}} \frac{U^2}{\gamma^{\frac{3}{2}} \mu_2^{\frac{1}{2}}}},$$

what does  $\langle \tilde{w}, \tilde{w} \rangle$  represent? What is the effect of the advection velocity  $U$  and viscosity  $\gamma$  on this coefficient?

Nota:  $\tilde{w}_n(x) = \xi_n H_n(\chi x) e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$  are all the adjoint eigenvectors.

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# Adjoint of linearized advection operator (1/4)

Theorem:

Let  $\mathcal{N}_{w_0} w = \begin{pmatrix} u \cdot \nabla u_0 + u_0 \cdot \nabla u \\ 0 \end{pmatrix}$  be an operator acting on  $w = (u, v, p)$  such that  $u = v = 0$  on boundaries. If

$\langle w_1, w_2 \rangle = \iint [u_1^* u_2 + v_1^* v_2 + p_1^* p_2] dx dy$ , the adjoint operator of  $\mathcal{N}_{w_0}$  is

$$\tilde{\mathcal{N}}_{w_0} = \begin{bmatrix} \partial_x u_0 & \partial_y u_0 & 0 \\ \partial_x v_0 & \partial_y v_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^* + \begin{bmatrix} -u_0^* \partial_x - v_0^* \partial_y & 0 & 0 \\ 0 & -u_0^* \partial_x - v_0^* \partial_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

# Adjoint of linearized advection operator (2/4)

$$\langle w_1, \mathcal{N}_{w_0} w_2 \rangle = \langle \tilde{\mathcal{N}}_{w_0} w_1, w_2 \rangle$$

$$\begin{aligned}
& \iint \left[ u_1^* (u_0 \partial_x u_2 + v_0 \partial_y u_2 + u_2 \partial_x u_0 + v_2 \partial_y u_0) \right. \\
& \quad \left. + v_1^* (u_0 \partial_x v_2 + v_0 \partial_y v_2 + u_2 \partial_x v_0 + v_2 \partial_y v_0) \right] dx dy \\
&= \iint \left[ (u_1^* u_0 \partial_x u_2 + u_1^* v_0 \partial_y u_2 + v_1^* u_0 \partial_x v_2 + v_1^* v_0 \partial_y v_2) \right. \\
& \quad \left. + (u_1^* \partial_x u_0 u_2 + u_1^* \partial_y u_0 v_2 + v_1^* \partial_x v_0 u_2 + v_1^* \partial_y v_0 v_2) \right] dx dy \\
&= \underbrace{\iint [u_1^* u_0 \partial_x u_2 + u_1^* v_0 \partial_y u_2 + v_1^* u_0 \partial_x v_2 + v_1^* v_0 \partial_y v_2]}_{(*)} dx dy \\
& \quad + \iint \left[ [u_1 \partial_x u_0^* + v_1 \partial_x v_0^*]^* u_2 + [u_1 \partial_y u_0^* + v_1 \partial_y v_0^*]^* v_2 \right] dx dy
\end{aligned}$$

# Adjoint of linearized advection operator (3/4)

$$\begin{aligned}
 (*) &= \overbrace{\int [u_1^* u_0 n_x u_2 + u_1^* v_0 n_y u_2 + v_1^* u_0 n_x v_2 + v_1^* v_0 n_y v_2] ds}^{0} \\
 &\quad - \iint [\partial_x(u_1^* u_0) u_2 + \partial_y(u_1^* v_0) u_2 + \partial_x(v_1^* u_0) v_2 + \partial_y(v_1^* v_0) v_2] dx dy \\
 &= - \iint \left[ [\partial_x(u_1 u_0^*) + \partial_y(u_1 v_0^*)]^* u_2 + [\partial_x(v_1 u_0^*) + \partial_y(v_1 v_0^*)]^* v_2 \right] dx dy
 \end{aligned}$$

$$\tilde{\mathcal{N}}_{w_0} w_1 = \begin{bmatrix} u_1 \partial_x u_0^* + v_1 \partial_x v_0^* \\ u_1 \partial_y u_0^* + v_1 \partial_y v_0^* \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} -\partial_x(u_1 u_0^*) - \partial_y(u_1 v_0^*) \\ -\partial_x(v_1 u_0^*) - \partial_y(v_1 v_0^*) \\ 0 \end{bmatrix}}_{\begin{bmatrix} -u_0^* \partial_x u_1 - v_0^* \partial_y u_1 \\ -u_0^* \partial_x v_1 - v_0^* \partial_y v_1 \\ 0 \end{bmatrix}}$$

# Adjoint of linearized advection operator (4/4)

Conclusion:

$$\begin{aligned}\tilde{\mathcal{N}}_{w_0} w_1 &= \begin{bmatrix} \partial_x u_0 & \partial_y u_0 & 0 \\ \partial_x v_0 & \partial_y v_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^* \begin{bmatrix} u_1 \\ v_1 \\ p_1 \end{bmatrix} + \begin{bmatrix} -u_0^* \partial_x - v_0^* \partial_y & 0 & 0 \\ 0 & -u_0^* \partial_x - v_0^* \partial_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ p_1 \end{bmatrix} \\ \mathcal{N}_{w_0} w_2 &= \begin{bmatrix} \partial_x u_0 & \partial_y u_0 & 0 \\ \partial_x v_0 & \partial_y v_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \\ p_2 \end{bmatrix} + \begin{bmatrix} u_0 \partial_x + v_0 \partial_y & 0 & 0 \\ 0 & u_0 \partial_x + v_0 \partial_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \\ p_2 \end{bmatrix}\end{aligned}$$

$\mathcal{N}_{w_0} \neq \tilde{\mathcal{N}}_{w_0}$  because of:

- component-type non-normality  $\Rightarrow v \rightarrow u$  becomes  $u \rightarrow v$
- convective-type non-normality  $\Rightarrow$  upstream convection

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# Adjoint of Stokes operator (1/4)

Theorem:

Let  $\mathcal{L} = \begin{pmatrix} -\nu\Delta() & \nabla() \\ -\nabla \cdot () & 0 \end{pmatrix}$  be an operator acting on  $w = (u, v, p)$  such that  $u = v = 0$  on boundaries. If  $\langle w_1, w_2 \rangle = \iint [u_1^* u_2 + v_1^* v_2 + p_1^* p_2] dx dy$ , the operator  $\mathcal{L}$  is self-adjoint :  $\tilde{\mathcal{L}} = \mathcal{L}$ .

# Adjoint of Stokes operator (2/4)

$$\langle w_1, \mathcal{L}w_2 \rangle = \langle \tilde{\mathcal{L}}w_1, w_2 \rangle \quad \mathcal{L} = \begin{pmatrix} -\nu\Delta() & \nabla() \\ -\nabla \cdot () & 0 \end{pmatrix}$$

$$\begin{aligned}
& \iint [u_1^*(-\nu\partial_{xx}u_2 - \nu\partial_{yy}u_2 + \partial_x p_2) + v_1^*(-\nu\partial_{xx}v_2 - \nu\partial_{yy}v_2 + \partial_y p_2) \\
& \quad + p_1^*(-\partial_x u_2 - \partial_y v_2)] dx dy \\
&= \iint [-\nu u_1^* \partial_{xx} u_2 - \nu u_1^* \partial_{yy} u_2 + u_1^* \partial_x p_2 - \nu v_1^* \partial_{xx} v_2 - \nu v_1^* \partial_{yy} v_2 \\
& \quad + v_1^* \partial_y p_2 - p_1^* \partial_x u_2 - p_1^* \partial_y v_2] dx dy \\
&= \int [-\nu \textcolor{red}{u}_1^* n_x \partial_x u_2 - \nu \textcolor{red}{u}_1^* n_y \partial_y u_2 + \textcolor{red}{u}_1^* n_x p_2 - \nu \textcolor{red}{v}_1^* n_x \partial_x v_2 - \nu \textcolor{red}{v}_1^* n_y \partial_y v_2 + \textcolor{red}{v}_1^* n_y p_2 \\
& \quad - p_1^* n_x \textcolor{red}{u}_2 - p_1^* n_y \textcolor{red}{v}_2] ds \\
& \quad - \iint [-\nu \partial_x u_1^* \partial_x u_2 - \nu \partial_y u_1^* \partial_y u_2 + \partial_x u_1^* p_2 - \nu \partial_x v_1^* \partial_x v_2 - \nu \partial_y v_1^* \partial_y v_2 \\
& \quad + \partial_y v_1^* p_2 - \partial_x p_1^* u_2 - \partial_y p_1^* v_2] dx dy
\end{aligned}$$

$\textcolor{red}{u} = \textcolor{red}{v} = 0$  on boundaries

# Adjoint of Stokes operator (3/4)

$$\begin{aligned}
 &= - \iint [\partial_x u_1^* p_2 + \partial_y v_1^* p_2 - \partial_x p_1^* u_2 - \partial_y p_1^* v_2] dx dy \\
 &\quad + \underbrace{\iint -[-\nu \partial_x u_1^* \partial_x u_2 - \nu \partial_y u_1^* \partial_y u_2 - \nu \partial_x v_1^* \partial_x v_2 - \nu \partial_y v_1^* \partial_y v_2] dx dy}_{(*)}
 \end{aligned}$$

$$\begin{aligned}
 (*) &= \int -[-\nu \partial_x u_1^* n_x \textcolor{red}{u}_2 - \nu \partial_y u_1^* n_y \textcolor{red}{u}_2 - \nu \partial_x v_1^* n_x \textcolor{red}{v}_2 - \nu \partial_y v_1^* n_y \textcolor{red}{v}_2] ds \\
 &\quad + \iint [-\nu \partial_{xx} u_1^* u_2 - \nu \partial_{yy} u_1^* u_2 - \nu \partial_{xx} v_1^* v_2 - \nu \partial_{yy} v_1^* v_2] dx dy
 \end{aligned}$$

$$\tilde{\mathcal{L}}_{W_1} = \begin{bmatrix} -\nu \partial_{xx} - \nu \partial_{yy} & 0 & \partial_x \\ 0 & -\nu \partial_{xx} - \nu \partial_{yy} & \partial_y \\ -\partial_x & -\partial_y & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ p_1 \end{bmatrix}$$

# Adjoint of Stokes operator (4/4)

$$\tilde{\mathcal{L}}w_1 = \begin{bmatrix} -\nu\partial_{xx} - \nu\partial_{yy} & 0 & \partial_x \\ 0 & -\nu\partial_{xx} - \nu\partial_{yy} & \partial_y \\ -\partial_x & -\partial_y & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ p_1 \end{bmatrix}$$

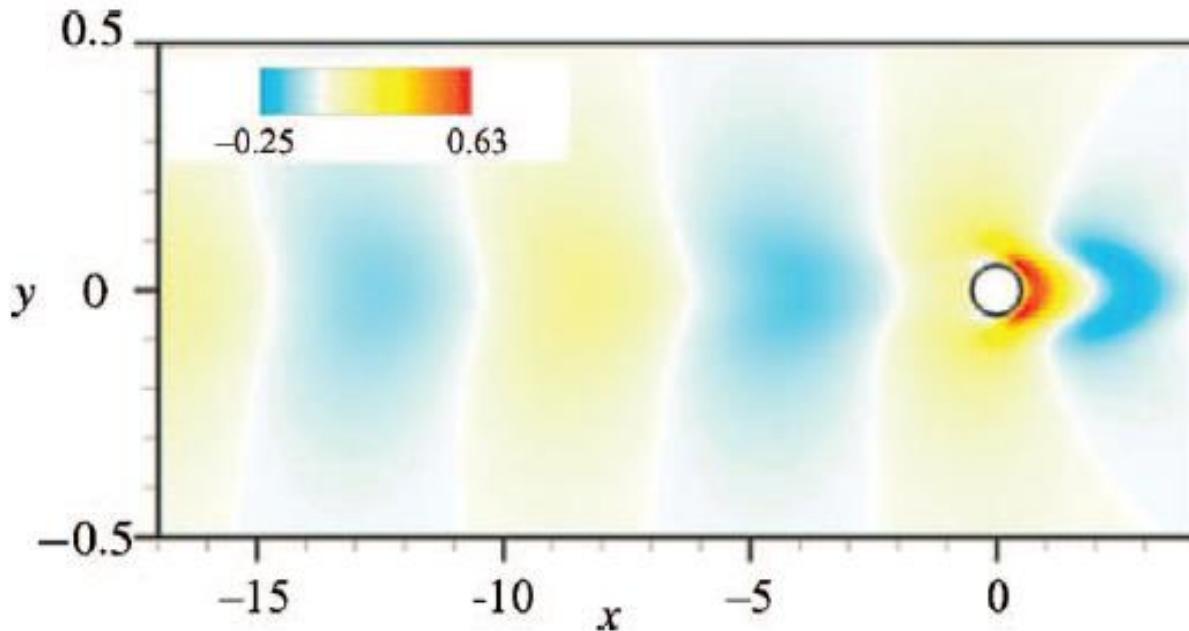
$$\mathcal{L}w_2 = \begin{bmatrix} -\nu\partial_{xx} - \nu\partial_{yy} & 0 & \partial_x \\ 0 & -\nu\partial_{xx} - \nu\partial_{yy} & \partial_y \\ -\partial_x & -\partial_y & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \\ p_2 \end{bmatrix}$$

$$\tilde{\mathcal{L}} = \mathcal{L}$$

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# The adjoint global mode of cylinder flow



Real part of cross-stream velocity field.  
Marginal adjoint global mode